ECE 313: Electromagnetic Waves

Lecture 5: Continue_ Maxwell’s equations

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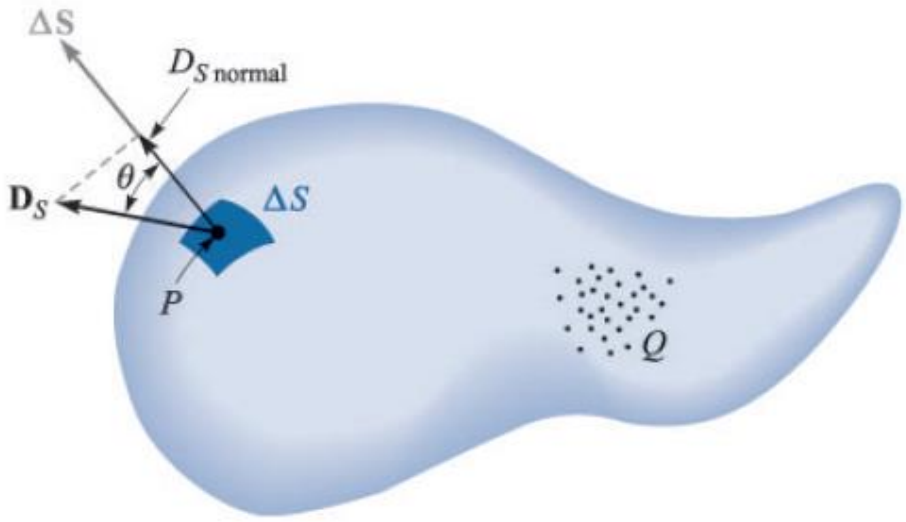
Gauss Law



- The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

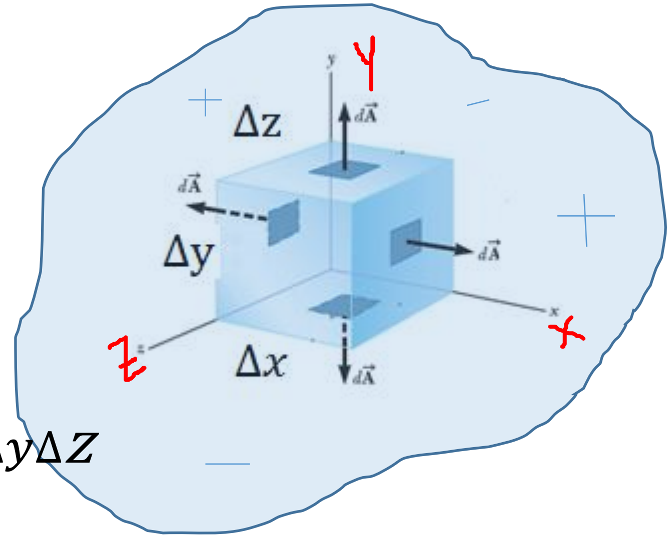
$$\oint_s D_s . ds = Q \ , \ Q = \int_v \rho_v \, dv$$

- Applying divergence theorem:

$$\oint_s D_s . ds = \int_v \nabla . D \, dv = \int_v \rho_v \, dv$$

$$\boxed{\nabla . D = \rho_v}$$

Gauss from integral to diff. form:



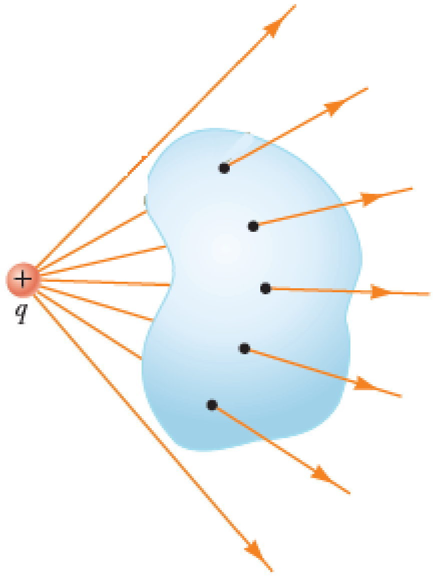
$$\oint_s D_s . ds = [D_x(x+\Delta x, y, z)\Delta y \Delta z - D_x(x,y,z)\ \Delta y \Delta z] + [D_y(x, y+\Delta y, z)\Delta x \Delta z - D_y(x,y,z)\ \Delta x \Delta z] + [D_z(x,y,z+\Delta z)\Delta x \Delta y - D_z(x,y,z)\ \Delta x \Delta y] = \rho_{v(x,y,z)\Delta x \Delta y \Delta Z}$$

Thus

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \frac{[D_x(x+\Delta x,y,z)-D_x(x,y,z)]}{\Delta x} + \frac{[D_y(x,y+\Delta y,z)-D_y(x,y,z)]}{\Delta y} + \frac{[D_z(x,y,z+\Delta Z)-D_z(x,y,z)]}{\Delta Z} = \rho_{v(x,y,z)}$$

$$\nabla . D = \rho_v$$

Case point charge located outside surface as in Fig.: The number of electric field lines entering the surface equals the number leaving the surface. Therefore, the net electric flux through a closed surface that surrounds no charge is zero.



$$\Delta\Psi = \text{flux crossing } \Delta\mathbf{S} = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

The *total* flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element $\Delta\mathbf{S}$,

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

Example

Giving a 60 μC point charge located at the origin , find the total electric flux passing through that portion of the sphere r=26 cm bounded by $0\leq \theta \leq \pi/2$ and $0\leq \varphi \leq \pi/2$

Ans: 60/8=7.5 μC

$$\textit{over all closed surface}$$

$$\oint_s D.ds = D * 4\pi r^2 = 60\mu \rightarrow D = \frac{60\mu}{4\pi r^2}$$

$$\textit{over specified surface}$$

$$\Delta\psi = \int_0^{\pi/2}\int_0^{\pi/2} D.r^2 \sin\theta d\theta d\varphi = \frac{60\mu}{4\pi r^2} r^2 * \frac{\pi}{2} * (-\cos\theta|_0^{\pi/2}) = \frac{60}{8} = 7.5\mu C$$

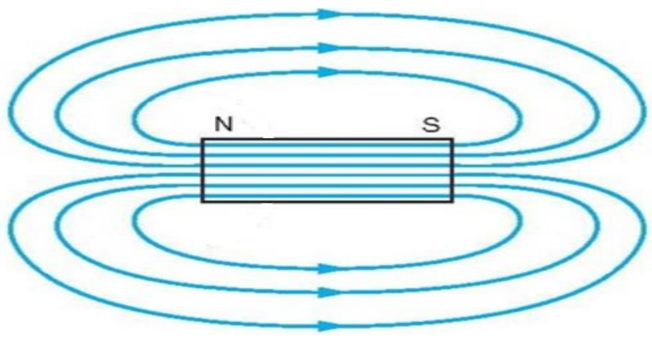
Guass' magnetic law

Differential form

$$\nabla . \underline{B} = 0$$

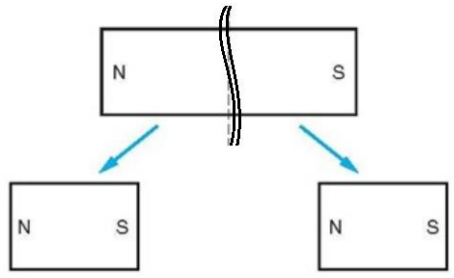
Integral form

$$\oint_s \underline{B}.\underline{ds} = 0$$



Divergence theorem $\int_v \nabla.\underline{B}dv = \oint_s \underline{B}.\underline{ds} = 0$

- The magnetic field lines are closed
- The magnetic field lines have neither source (start point) or sink (end point)
- It is not possible to have an isolated magnetic poles (or magnetic charges)
- Magnetic field is not a flow source, but a solenoidal source



Maxwell's equations

Differential form	Integral form	
$\nabla \times \bar{E} = -\frac{\partial B}{\partial t}$	$\oint_c \bar{E}.dl = -\frac{d}{dt}\int_s \bar{B}.ds$	**Faraday's law of induction**
$\nabla \times \bar{H} = \bar{J} + \frac{\partial D}{\partial t}$	$\oint_c \bar{H}.\overline{dl} = \int_s \bar{J}.\overline{ds} + \frac{d}{dt}\int_s \bar{D}.ds$	**Ampere's law**
$\nabla.\bar{D} = \rho$	$\oint_s \bar{D}_s.ds = \int_v \rho_v\, dv$	**Gauss flux theorem**
$\nabla.\bar{B} = 0$	$\oint_s \underline{B}.\underline{ds} = 0$	**Magnetic flux conservation**
$\nabla.\underline{J}_c = -\frac{d\rho_v}{dt}$	$\oint_s \underline{J}_c.\underline{ds} = -\frac{d}{dt}\int_v \rho_v dv$	**Continuity equation**

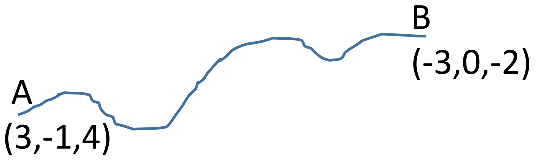
Conservative and solenoidal vector fields

- Conservative field (irrotational): it is the vector field that can be expressed as gradient of other scalar. Line integral of conservative field is path independent. *curl of conservative field=0 (as curl grad=0)*

In **electrostatic**:

- $\bar{E}$ is a conservative field as $\bar{E} = -\nabla V$, and $\int \bar{E}.d\bar{l}$ does not depend on path (it depend on start and end points of integrals), thus closed contour integral (end point same as start point) for conservative field=0. $\oint_c \bar{E}.d\bar{l} = 0$
- V is called potential function of $\bar{E}$

Example: for conservative field $A(\bar{r}) = \nabla\left(x^2 + y^2\right)z$ find $\int_A^B A(\bar{r}).\overline{dl}$



$$g_A = \left((3)^2 + (-1)^2\right)4 = 40 \qquad g_B = \left((-3)^2 + (0)^2\right)(-2) = -18$$

$$\int_C \vec{A}(\bar{r}).\overline{dl} = \int_C \nabla g(\bar{r}).\overline{dl} = -18 - 40 = -58$$

Gradient theorem

$$\int_{\mathbf{p}}^{\mathbf{q}} \nabla\varphi(\mathbf{r}) \cdot d\mathbf{r} = \varphi(\mathbf{q}) - \varphi(\mathbf{p})$$

$$\nabla f = df / dl\ \hat{a}_{dl} \rightarrow$$

$$\int \nabla f . dl\ \hat{a}_{dl} = df = f2 - f1$$

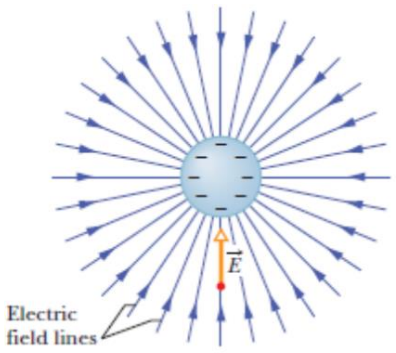
Conservative and solenoidal vector fields

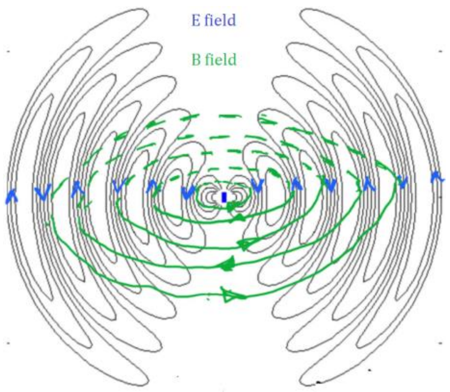
Curl grad=0
Div curl=0

- Every solenoidal vector field can be expressed as the curl of some other vector field.(The curl of **any** vector field **always** results in a solenoidal field)
- only solenoidal vector $\overline{B}$ have zero divergence. $\nabla.\overline{B}=0$ Solenoidal field = divergenceless.
- the **surface** integral of **any** and **every** solenoidal vector field across a **closed** surface is equal to zero (since divergence theorem $\int_v \nabla.\overline{B}dv = \oint_s \overline{B}.d\overline{s} = 0$)
- In points:

1. ***Every*** *solenoidal field can be expressed as the* ***curl*** *of some* ***other*** *vector field.*
2. *The curl of* ***any*** *and* ***all*** *vector fields always results in a solenoidal vector field.*
3. *The* ***surface integral*** *of a solenoidal field across any* ***closed*** *surface is equal to* ***zero.***
4. *The* ***divergence*** *of every solenoidal vector field is equal to* ***zero****.*
5. *The divergence of a vector field is zero* ***only*** *if it is* ***solenoidal***

Example of Conservative and solenoidal vector fields: Electric field in electrostatic and time varying:





In electrostatic

Electric field is conservative

Vector field

In time varying

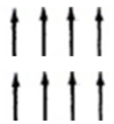
Electric field produced by changing

In magnetic field

Is solenoid vector field

If curl of a field is zero it is conservative vector field ***If div of a field is zero it is solenoidal vector field***

Types of vector fields



$\nabla X \bar{A} = 0$

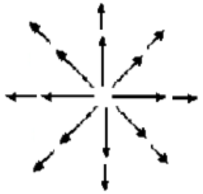
$\nabla \cdot \bar{A} = 0$

Uniform field

$\nabla X \bar{A} \neq 0$

$\nabla \cdot \bar{A} = 0$

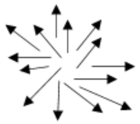
solenoidal (rotational) field



$\nabla X \bar{A} = 0$

$\nabla \cdot \bar{A} \neq 0$

Irrotational (conservative) field



$\nabla X \bar{A} \neq 0$

$\nabla \cdot \bar{A} \neq 0$

General field

Potential functions (in time varying fields)

As B is solenoidal vector, a vector magnetic potential A can be defined from B

$$\nabla.\overline{B} = 0 \quad \rightarrow \overline{B} = \nabla \times \overline{A}$$

From faraday's law

$$\nabla \times \overline{E} = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \overline{A}) \quad \rightarrow \quad \nabla \times (\overline{E} + \frac{\partial \overline{A}}{\partial t}) = 0$$

$$\overline{E} + \frac{\partial \overline{A}}{\partial t} = -\nabla V \ , \boxed{\overline{E} = -\frac{\partial \overline{A}}{\partial t} - \nabla V} \quad in\ static\ \frac{\partial}{\partial t} = 0\ and\ \overline{E} = -\nabla V$$

Time varying magnetic field (→ $-\frac{\partial \overline{A}}{\partial t}$) Accumulation of charges (→ $-\nabla V$)

$$\overline{A} = \frac{\mu}{4\pi}\int_v \frac{\overline{J}e^{-jKR}}{R} dv \qquad V = \frac{1}{4\pi\varepsilon}\int_v \frac{\rho e^{-jkR}}{R} dv$$

Potential functions (in time varying fields)

- *Non homogeneous wave equation for vector potential A*

:$$\nabla \times \overline{H} = (\frac{1}{\mu}\nabla \times \nabla \times \overline{A}) = \frac{\partial \overline{D}}{\partial t} + \overline{J}$$

$$\nabla \times \nabla \times \overline{A} = \mu\varepsilon \frac{\partial \overline{E}}{\partial t} + \mu\overline{J}$$

$$\nabla \times \nabla \times \overline{A} = \mu\varepsilon \frac{\partial}{\partial t}(-\nabla V - \frac{\partial \overline{A}}{\partial t}) + \mu\overline{J}$$

$$\nabla(\nabla.\overline{A}) - \nabla^2 \overline{A} = -\nabla(\mu\varepsilon \frac{\partial V}{\partial t}) - \mu\varepsilon \frac{\partial^2 \overline{A}}{\partial t^2} + \mu\overline{J} \quad \textit{vector defined by its curl and its divergence}$$

$$let \quad \nabla.\overline{A} = -\mu\varepsilon \frac{\partial V}{\partial t}$$ → ***Lorentz condition for potential***

$$\nabla^2 \overline{A} - \mu\varepsilon \frac{\partial^2 \overline{A}}{\partial t^2} = -\mu\overline{J}$$

wave equation, its solution represent waves travelling with velocity= $1/\sqrt{\mu\varepsilon}$

Potential functions (in time varying fields)

Non homogeneous wave equation for scalar potential V:

$$\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t}, \quad \nabla.\bar{D} = \rho$$

$$\nabla.\varepsilon(-\nabla V - \frac{\partial \bar{A}}{\partial t}) = \rho \quad \rightarrow \quad -\nabla^2 V - \frac{\partial}{\partial t}(\nabla.\bar{A}) = \rho / \varepsilon$$

$$\nabla.\bar{A} = -\mu\varepsilon \frac{\partial V}{\partial t} \quad \rightarrow \quad \boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\rho / \varepsilon}$$

wave equation

$$\nabla^2 V + k^2 V = -\rho / \varepsilon$$

TIME-HARMONIC ELECTROMAGNETICS

$$\mathbf{E}(x, y, z, t) = \mathcal{R}e[\mathbf{E}(x, y, z)e^{j\omega t}],$$

$$\nabla^2 V + k^2 V = -\frac{\rho}{\epsilon}$$

$$\boxed{V(R) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho e^{-jkR}}{R}\, dv' \qquad \text{(V)},}$$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J},$$

$$\boxed{\mathbf{A}(R) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J} e^{-jkR}}{R}\, dv' \qquad \text{(Wb/m).}}$$

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{u}$$

The formal procedure for determining the electric and magnetic fields due to time-harmonic charge and current distributions is as follows:†

1. Find phasors $V(R)$ and $\mathbf{A}(R)$ from Eqs. (7–99) and (7–100).
2. Find phasors $\mathbf{E}(R) = -\nabla V - j\omega\mathbf{A}$ and $\mathbf{B}(R) = \nabla \times \mathbf{A}$.
3. Find instantaneous $\mathbf{E}(R, t) = \mathcal{R}e[\mathbf{E}(R)e^{j\omega t}]$ and $\mathbf{B}(R, t) = \mathcal{R}e[\mathbf{B}(R)e^{j\omega t}]$ for a cosine reference.

The degree of difficulty of a problem depends on how difficult it is to perform the integrations in Step 1.

Example

In a region where $\mu_R = \epsilon_R = 1$ and $\sigma = 0$, the retarded potentials are given by $V = x(z - ct)$ V and $\mathbf{A} = x[(z/c) - t]\mathbf{a}_z$ Wb/m, where $c = 1/\sqrt{\mu_0\epsilon_0}$.

a) Show that $\nabla \cdot \mathbf{A} = -\mu\epsilon(\partial V/\partial t)$:

First,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = \frac{x}{c} = x\sqrt{\mu_0\epsilon_0}$$

Second,

$$\frac{\partial V}{\partial t} = -cx = -\frac{x}{\sqrt{\mu_0\epsilon_0}}$$

b) Find **B**, **H**, **E**, and **D**:

Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_x}{\partial x}\mathbf{a}_y = \underline{\left(t - \frac{z}{c}\right)\mathbf{a}_y \text{ T}}$$

Then

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{1}{\mu_0}\left(t - \frac{z}{c}\right)\mathbf{a}_y \text{ A/m}$$

Now,

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -(z - ct)\mathbf{a}_x - x\mathbf{a}_z + x\mathbf{a}_z = \underline{(ct - z)\mathbf{a}_x \text{ V/m}}$$

Then

$$\mathbf{D} = \epsilon_0\mathbf{E} = \underline{\epsilon_0(ct - z)\mathbf{a}_x \text{ C/m}^2}$$

Near field:
Propagation does not appear through A and V it appears
In E and H it is along z
E along x
B along y
E not as same direction of A since it depend on both grad V and A
-in far field E will depend on A only)